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A Higher-Order Discretized Algorithm for Solving Quadratic Optimal Control Problems

Monica Veronica Crankson [1], Olusegun Olotu [2] Newton Amegbey [3]

[1] Department of Mathematical Sciences, University of Mines and Technology, Tarkwa - Ghana
[2] Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State - Nigeria
[3] Mining Engineering Department, University of Mines and Technology, Tarkwa - Ghana

ABSTRACT
This research proposes an algorithm for the solutions of quadratic optimal control problems with ordinary differentialequation constraint using higher-order discretization schemes. The objective functional and the constraints are discretized using4th order Simpson's 3/8 rule and 6th order Adams Moulton's method respectively. The Quadratic Penalty Function methodis used to obtain the unconstrained formulations of the discretized constrained problems which gives a framework for the useof the Conjugate Gradient Method. Two examples are considered and the solutions from the new algorithm presented. Thenew algorithm is established to be effective, efficient, robust and accurate as it converges faster and favorably to the analyticalsolution.
Keywords - Optimal Control, Discretization, Quadratic Penalty Function Method, Conjugate Gradient Method, Converge.

I. INTRODUCTION

Optimization and optimal control pervade mathematicsand science as they are the main tools in decision making.Research in these areas is accelerating at a rapid pace dueto their numerous applications in various disciplines [1].Optimal control and its applications are found in diverse fields,including aerospace, robotics, engineering, biomedical sciences,economics, finance and management science, and it continuesto be an active area of interest in control theory [2].
Optimization is the process in which the best feasible solutionto a problem is found. This involves finding an extremumof some functions [3]. In solving optimization problems,algorithms that end up in a finite number of steps, or iterativemethods that converge to a solution (in some specific class ofproblems) can be used. Heuristics that provide approximatesolutions to some problems, although their iterations do notnecessarily converge can also be considered [6].
Before one can optimize an objective, a quantitative measureof the performance of the system must be identified. Thisobjective could be profit, time, potential energy, quantity or acombination of quantities that can be represented by a singlenumber. The objective depends on certain characteristics ofthe system termed as variables. The goal is to find the valuesof the variables that optimize the objective. These variablesare often restricted or constrained in a way [5].
Optimization problems are categorized as constrained andunconstrained. Constrained optimization problems arise frommodels in which constraints play an essential role, for example,imposing shape constraints in a design problem. Unconstrainedoptimization problems on the other hand, arise directly in manypractical applications, where an objective function is optimizedwith no restrictions on these variables [7].
The presence of constraints creates more challenges whilefinding the optimum than the unconstrained problems sinceone needs to find points that satisfy all the constraints. Oneapproach in solving such problem is to reformulate theconstrained problem as an unconstrained problem by replacingthe constraints with penalization terms and adding to theobjective function depending on the number of constraintsviolated. The penalty function to be determined vary from oneproblem to another, however these penalties should satisfy allthe constraints at the end [4].
Optimal control deals with finding the control and statevariables to a dynamical system over a period of time tooptimize a specified performance index while satisfying anyconstraints on the motion. As such, an Optimal Control Problem(OCP) requires a performance index or a cost functional whichis a function of the state and control variables. Its main goalis to find a piecewise continuous control and the associatedstate variable that optimize a given objective functional [8].
Generally, an optimal control problem is considered as anoptimization problem, even though there is a difference in theoptimizer. The optimizer in optimal control theory is not justa single value, but a function called the optimal control [9].A constrained dynamic continuous optimal control problem isgenerally defined as

$$\text{Minimize } J(x(t),u(t)) = \int_{t_0}^{t_f} f(t,x(t),u(t))dt \quad (1)$$

$$\text{Subject to } \quad \dot{x}(t) = h(t,x(t),u(t)) \quad (2)$$

$$x(t_0) = x_0, \quad t_0 \le t \le t_f \quad (3)$$

where t represents the independent time variable, t0 and tf arethe initial and terminal times respectively, x(t) 2 <n is a vectorof state variables and u(t) 2 <m is a vector of control variables2which are going to be optimized, f : < _ <n _ <m ! < isthe functional and h : <_<n _<m ! <p is a smooth vectorfield. Both f and h are continuously differentiable functions,that is, f 2 C2[t0; tf] and h 2 C1[t0; tf]. x0 is the knowninitial state and the final state x(tf) could be free (unrestricted)or fixed (x(tf) = xf).

There are two major classes of numerical methods forsolving optimal control problems, namely the direct andindirect methods. In a direct method, the state and/or controlvariables is discretized on a time grid using some formof collocation method. This transforms the problem to anonlinear optimization problem or nonlinear programmingproblem (NLP). The resulting nonlinear programming problemis then solved using various established NLP packages [6].The complete discretization of the state and control functionseliminate the need to iteratively solve the initial value problem(IVP) although this may lead to a large number of decisionvariables for the NLP solver [10]. Partial parametrization ofthe control functions is also used in other direct approachesby using a piecewise constant or higher order polynomialapproximations [7].

Sargent [11] presented a review on the different numericalapproaches to the solutions of optimal control problems and abrief historical survey of the development of optimal controland calculus of variations. The least square method was alsoused to obtain numerical solutions to linear quadratic optimalcontrol problems based on Bezier control points which providesa bound on the residual function. The examples consideredshowed that the approximate functions are satisfactory for alarger step size [12].

The work of [13] examined the analytical and numericalsolutions of optimal control problems with vector-matrixcoefficients. Variational iteration method was incorporatedto solve the resulting general riccati differential equation.The results showed that both the analytical and numericalsolutions agreed favourably. A computational method based onstate parametrization was presented to solve optimal controlproblems. The state variables were approximated by Boubakerpolynomials with unknown coefficients while the equationof motion, performance index and boundary conditions wereconverted into some algebraic equations. This gave rise to anoptimization problem which is easily solved by establishedmethods. Examples were solved to demonstrate the applicabilityand efficiency of the method [14].

An embedding method was introduced by [15] to findapproximate solutions to nonlinear optimal control problemswith mixed constraints which have delays in both the state andcontrol variables. The solutions were obtained from solving thecorresponding finite dimensional linear programming problem.Hypothetical examples were used to illustrate the effectivenessand applicability of the proposed idea. Olotu and Dawodu [16]developed a Quasi-Newton embedded augmented Lagrangianalgorithm for delay proportional optimal control problems usingthe ”first discretize and optimize” approach. The delay termswere also discretized over the entire delay interval to ensureits piecewise continuity at each grid point.

A practical spreadsheet method was recently introduced tosolve a class of optimal control problems. Two elementary calculus functions were utilized in this method, that is, an IVPsolver and a discrete data integrator from Excel calculus Add-in.These functions were used together with Excel intrinsic NLPsolver to formulate a partial-parametrization direct solutionstrategy. A cost index was represented by an equivalent formulathat fully encapsulated a control-parametrized inner IVP byuse of the calculus functions. The Excel NLP solver was usedto optimize the cost index by varying a decision parametervector, subject to bound constraints on the state and controlvariables [17].

An extension was made to more general formulations ofoptimal control in another research which demonstrated a systematicsolution strategy based on an adaptation of the partialparametrizationdirect solution method. This preserves thestructure of the original mathematical optimization statement,and transforms it into a simplified NLP problem suitable forExcel NLP solver. This NLP Solver Command is based onthe Generalized Reduced Gradient Method (GRG) which iscompatible with the calculus functions. The convergence anderror control of the method was investigated, and comparedfavorably with published solutions obtained by fundamentallydifferent methods [18].

The analytical solutions of optimal control problems withmixed constraints were examined by [19]. This was obtainedby applying the first order optimality conditions on theHamiltonian function and solving the resulting system of firstorder ordinary differential equations. This led to the optimalstate, control and adjoint variables and hence the optimalobjective function value. The approach was used to solvesome examples.

Optimal control problems constrained by Ordinary DifferentialEquations (ODEs) has a lot of applications in engineering,economics, biology and medicine but are becoming too complexto solve analytically due to the complexity nature of differentreal-life problems around us. There is therefore the need todevelop algorithms with numerical solutions very close tothe analytical solution and with a faster rate of convergencecompared to existing algorithms. Most existing algorithmsare often based on approximating linear search parameterin optimizing the problem, or developing rigorous controloperator which is cumbersome in structure. To avoid these numerouscomputations, a discretized continuous algorithm witha constructed operator by the use of quadratic programming isproposed.

II. METHODOLOGY

The general quadratic optimal control problems are a classof optimal control problems whose cost functional is quadratic,and they arise in a wide range of applications. Of a specialinterest is the general quadratic optimal control problemformulated as

$$\begin{aligned} \text{Minimize } J(x,u) = &\int_0^T (ax^2(t)+bu^2(t))dt \\ \text{Subject to } \dot{x}(t) = & \, cx(t)+du(t) \\ & x(0)=x_0, \quad t\in[0,T] \qquad (4) \end{aligned}$$

$a,b,c,d \in \Re;\ a,b>0$ where $x(t)$ is the state variable which describes the system and $u(t)$ is the control variable which directs the system.

A. Discretization of the Objective Functional

$$J(x,u) = \int_0^T (ax^2(t)+bu^2(t))dt \qquad (5)$$

$$= \int_0^T ax^2(t)dt + \int_0^T bu^2(t)dt \qquad (6)$$

We proceed by discretizing the objective functional using the Fourth-Order Simpson's $\frac{3}{8}$ Rule given as

$$\int_{t_0}^{t_n} f(t)dt = \frac{3h}{8}\left[f(t_0)+3\sum_{i=1,4,7,\dots}^{n-2} f(t_i)+3\sum_{i=2,5,8,\dots}^{n-1} f(t_i) + 2\sum_{i=3,6,9,\dots}^{n-3} f(t_i)+f(t_n)\right] \qquad (7)$$

Defining $n=\frac{t_n-t_0}{h}=\frac{T-0}{h}=\frac{T}{h}$, where n is the number of partitions and h is the step size, we have

$$J(x,u) = a\left(\frac{3h}{8}\right)\left[f(0)+3\sum_{i=1,4,7,\dots}^{n-2} x_i^2+3\sum_{i=2,5,8,\dots}^{n-1} x_i^2+ 2\sum_{i=3,6,9,\dots}^{n-3} x_i^2+f(n)\right]+b\left(\frac{3h}{8}\right)\left[f(0)+ 3\sum_{i=1,4,7,\dots}^{n-2} u_i^2+3\sum_{i=2,5,8,\dots}^{n-1} u_i^2+2\sum_{i=3,6,9,\dots}^{n-3} u_i^2+ f(n)\right] \qquad (8)$$

$$J(x,u) = \frac{3ah}{8}\left[x_0^2+3\sum_{i=1,4,7,\dots}^{n-2} x_i^2+3\sum_{i=2,5,8,\dots}^{n-1} x_i^2+ 2\sum_{i=3,6,9,\dots}^{n-3} x_i^2+x_n^2\right]+\frac{3bh}{8}\left[u_0^2+3\sum_{i=1,4,7,\dots}^{n-2} u_i^2+ 3\sum_{i=2,5,8,\dots}^{n-1} u_i^2+2\sum_{i=3,6,9,\dots}^{n-3} u_i^2+u_n^2\right] \qquad (9)$$

Setting $\frac{M}{2}=\frac{3ah}{8}$ and $\frac{N}{2}=\frac{3bh}{8}$ in (10) gives

$$J(x,u) = \frac{M}{2}\left[x_0^2+3(x_1^2+x_4^2+x_7^2+\dots+x_{n-2}^2)+3(x_2^2 + x_5^2+x_8^2+\dots+x_{n-1}^2)+2(x_3^2+x_6^2+x_9^2+\dots+ x_{n-3}^2)+x_n^2\right]+\frac{N}{2}\left[u_0^2+3(u_1^2+u_4^2+u_7^2+\dots+ u_{n-2}^2)+3(u_2^2+u_5^2+u_8^2+\dots+u_{n-1}^2)+2(u_3^2+ u_6^2+u_9^2+\dots+u_{n-3}^2)+u_n^2\right] \qquad (11)$$

$$J(x,u) = \frac{M}{2}\left[x_0^2+3x_1^2+3x_2^2+2x_3^2+3x_4^2+3x_5^2+2x_6^2 +\dots+2x_{n-3}^2+3x_{n-2}^2+3x_{n-1}^2+x_n^2\right]+\frac{N}{2}\left[u_0^2 +3u_1^2+3u_2^2+2u_3^2+3u_4^2+3u_5^2+2u_6^2+\dots+ 2u_{n-3}^2+3u_{n-2}^2+3u_{n-1}^2+u_n^2\right] \qquad (12)$$

In matrix form, we have

$$J(X,U) = \frac{M}{2}x_0^2+\frac{1}{2}X^TAX+\frac{1}{2}U^TBU \qquad (13)$$

where $A\in\Re^{n\times n},\ B\in\Re^{(n+1)\times(n+1)},\ X\in\Re^n,\ U\in\Re^{n+1}$.

The Augmentation of the matrix of the state and control variables is given as

$$W = [X \mid U] \in \Re^{2n+1} \qquad (14)$$

and

$$P = \begin{pmatrix} A & | & 0 \\ 0 & | & B \end{pmatrix} \in \Re^{(2n+1)\times(2n+1)} \qquad (15)$$

where

$$W = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \\ u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

and

$$P = \begin{bmatrix}
3M & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 3M & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 2M & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 2M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 3M & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 3M & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & N & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 3N & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 3N & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2N & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2N & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3N & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3N & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & N
\end{bmatrix}$$

In a more concise vector-matrix form, we have

$$J(W) = \frac{1}{2}W^T PW + F \qquad (16)$$

where F is the constant $\frac{M}{2}x_0^2$.

The dimensional coefficient matrix P is expressed compactly as

$$P = [p_{ij}] = \begin{cases} 3M, & i=j,\ i=1,4,7,...,n-2; \\ 3M, & i=j,\ i=2,5,8,...,n-1; \\ 2M, & i=j,\ i=3,6,9,...,n-3; \\ M, & i=j=n; \\ N, & i=j=n+1, 2n+1; \\ 3N, & i=j=n+2, n+5, ..., 2n-1; \\ 3N, & i=j=n+3, n+6, ..., 2n; \\ 2N, & i=j=n+4, n+7, ..., 2n-2; \\ 0, & i \neq j. \end{cases}$$

B. Discretization of the Constraint

$$\dot{x}(t) = cx(t) + du(t) \qquad (17)$$

The constraint is discretized using the Sixth-Order Adams-Moulton Method which is defined as

$$x_{i+5} = x_{i+4} + \frac{h}{1440}[475f_{i+5} + 1427f_{i+4} - 798f_{i+3} + 482f_{i+2} - 173f_{i+1} + 27f_i] \qquad (18)$$

Applying the Adams-Moulton Technique, we have

$$x_{i+5} = x_{i+4} + \frac{ch}{1440}\Big[(475x_{i+5} + 1427x_{i+4} - 798x_{i+3} + 482x_{i+2} - 173x_{i+1} + 27x_i)\Big] + \frac{dh}{1440}\Big[(475u_{i+5} + 1427u_{i+4} - 798u_{i+3} + 482u_{i+2} - 173u_{i+1} + 27u_i)\Big] \qquad (19)$$

$$x_{i+5} = x_{i+4} + \frac{475ch}{1440}x_{i+5} + \frac{1427ch}{1440}x_{i+4} - \frac{798ch}{1440}x_{i+3} + \frac{482ch}{1440}x_{i+2} - \frac{173ch}{1440}x_{i+1} + \frac{27ch}{1440}x_i + \frac{475dh}{1440}u_{i+5} + \frac{1427dh}{1440}u_{i+4} - \frac{798dh}{1440}u_{i+3} + \frac{482dh}{1440}u_{i+2} - \frac{173dh}{1440}u_{i+1} + \frac{27dh}{1440}u_i \qquad (20)$$

Multiplying through by 1440 and grouping like terms yields

$$(1440 - 475ch)x_{i+5} = (1440 + 1427ch)x_{i+4} - 798chx_{i+3} + 482chx_{i+2} - 173chx_{i+1} + 27chx_i + 475dhu_{i+5} + 1427dhu_{i+4} - 798dhu_{i+3} + 482dhu_{i+2} - 173dhu_{i+1} + 27dhu_i \qquad (21)$$

Dividing through by $1440 - 475ch$ gives

$$x_{i+5} = \frac{1440 + 1427ch}{1440 - 475ch}x_{i+4} - \frac{798ch}{1440 - 475ch}x_{i+3} + \frac{482ch}{1440 - 475ch}x_{i+2} - \frac{173ch}{1440 - 475ch}x_{i+1} + \frac{27ch}{1440 - 475ch}x_i + \frac{475dh}{1440 - 475ch}u_{i+5} + \frac{1427dh}{1440 - 475ch}u_{i+4} - \frac{798dh}{1440 - 475ch}u_{i+3} + \frac{482dh}{1440 - 475ch}u_{i+2} - \frac{173dh}{1440 - 475ch}u_{i+1} + \frac{27dh}{1440 - 475ch}u_i \qquad (22)$$

Let

$$v_1 = \frac{1440 + 1427ch}{1440 - 475ch}; \quad v_2 = \frac{-798ch}{1440 - 475ch};$$
$$v_3 = \frac{482ch}{1440 - 475ch}; \quad v_4 = \frac{-173ch}{1440 - 475ch};$$
$$v_5 = \frac{27ch}{1440 - 475ch}; \quad w_1 = \frac{475dh}{1440 - 475ch};$$
$$w_2 = \frac{1427dh}{1440 - 475ch}; \quad w_3 = \frac{-798dh}{1440 - 475ch};$$
$$w_4 = \frac{482dh}{1440 - 475ch}; \quad w_5 = \frac{-173dh}{1440 - 475ch};$$
$$w_6 = \frac{27dh}{1440 - 475ch}$$

Equation (22) becomes

$$x_{i+5} = v_1x_{i+4} + v_2x_{i+3} + v_3x_{i+2} + v_4x_{i+1} + v_5x_i + w_1u_{i+5} + w_2u_{i+4} + w_3u_{i+3} + w_4u_{i+2} + w_5u_{i+1} + w_6u_i \qquad (23)$$

which is known as the recurrence relation for $i = 0, 1, 2, \cdots, n-5$.
At $i = 0$,

$$x_5 = v_1x_4 + v_2x_3 + v_3x_2 + v_4x_1 + v_5x_0 + w_1u_5 + w_2u_4 + w_3u_3 + w_4u_2 + w_5u_1 + w_6u_0$$
$$\Rightarrow -v_4x_1 - v_3x_2 - v_2x_3 - v_1x_4 + x_5 - w_6u_0 - w_5u_1 - w_4u_2 - w_3u_3 - w_2u_4 - w_1u_5 = v_5x_0 \qquad (24)$$

At $i = 1$,

$$x_6 = v_1x_5 + v_2x_4 + v_3x_3 + v_4x_2 + v_5x_1 + w_1u_6 + w_2u_5 + w_3u_4 + w_4u_3 + w_5u_2 + w_6u_1$$

$$\Rightarrow -v_5x_1 - v_4x_2 - v_3x_3 - v_2x_4 - v_1x_5 + x_6 - w_6u_1 - w_5u_2 - w_4u_3 - w_3u_4 - w_2u_5 - w_1u_6 = 0 \tag{25}$$

At $i = 2$,

$$x_7 = v_1x_6 + v_2x_5 + v_3x_4 + v_4x_3 + v_5x_2 + w_1u_7 + w_2u_6 + w_3u_5 + w_4u_4 + w_5u_3 + w_6u_2$$

$$\Rightarrow -v_5x_2 - v_4x_3 - v_3x_4 - v_2x_5 - v_1x_6 + x_7 - w_6u_2 - w_5u_3 - w_4u_4 - w_3u_5 - w_2u_6 - w_1u_7 = 0 \tag{26}$$

At $i = 3$,

$$x_8 = v_1x_7 + v_2x_6 + v_3x_5 + v_4x_4 + v_5x_3 + w_1u_8 + w_2u_7 + w_3u_6 + w_4u_5 + w_5u_4 + w_6u_3$$

$$\Rightarrow -v_5x_3 - v_4x_4 - v_3x_5 - v_2x_6 - v_1x_7 + x_8 - w_6u_3 - w_5u_4 - w_4u_5 - w_3u_6 - w_2u_7 - w_1u_8 = 0 \tag{27}$$

$\vdots \qquad\qquad \vdots$

At $i = n-7$,

$$x_{n-2} = v_1x_{n-3} + v_2x_{n-4} + v_3x_{n-5} + v_4x_{n-6} + v_5x_{n-7} + w_1u_{n-2} + w_2u_{n-3} + w_3u_{n-4} + w_4u_{n-5} + w_5u_{n-6} + w_6u_{n-7}$$

$$\Rightarrow -v_5x_{n-7} - v_4x_{n-6} - v_3x_{n-5} - v_2x_{n-4} - v_1x_{n-3} + x_{n-2} - w_6u_{n-7} - w_5u_{n-6} - w_4u_{n-5} - w_3u_{n-4} - w_2u_{n-3} - w_1u_{n-2} = 0 \tag{28}$$

At $i = n-6$,

$$x_{n-1} = v_1x_{n-2} + v_2x_{n-3} + v_3x_{n-4} + v_4x_{n-5} + v_5x_{n-6} + w_1u_{n-1} + w_2u_{n-2} + w_3u_{n-3} + w_4u_{n-4} + w_5u_{n-5} + w_6u_{n-6}$$

$$\Rightarrow -v_5x_{n-6} - v_4x_{n-5} - v_3x_{n-4} - v_2x_{n-3} - v_1x_{n-2} + x_{n-1} - w_6u_{n-6} - w_5u_{n-5} - w_4u_{n-4} - w_3u_{n-3} - w_2u_{n-2} - w_1u_{n-1} = 0 \tag{29}$$

At $i = n-5$,

$$x_n = v_1x_{n-1} + v_2x_{n-2} + v_3x_{n-3} + v_4x_{n-4} + v_5x_{n-5} + w_1u_n + w_2u_{n-1} + w_3u_{n-2} + w_4u_{n-3} + w_5u_{n-4} + w_6u_{n-5}$$

$$\Rightarrow -v_5x_{n-5} - v_4x_{n-4} - v_3x_{n-3} - v_2x_{n-2} - v_1x_{n-1} + x_n - w_6u_{n-5} - w_5u_{n-4} - w_4u_{n-3} - w_3u_{n-2} - w_2u_{n-1} - w_1u_n = 0 \tag{30}$$

Taking equations (24) to (30), the discretized form of the constraints can be presented in matrix form.

The matrix of the **state variable** is given as

$$\begin{bmatrix} -v_4 & -v_3 & -v_2 & -v_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ -v_5 & -v_4 & -v_3 & -v_2 & -v_1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -v_5 & -v_4 & -v_3 & -v_2 & -v_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -v_5 & -v_4 & -v_3 & -v_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_5 & -v_4 & -v_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -v_5 & -v_4 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -v_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -v_2 & -v_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -v_3 & -v_2 & -v_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -v_4 & -v_3 & -v_2 & -v_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -v_5 & -v_4 & -v_3 & -v_2 & -v_1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \vdots \\ x_{n-5} \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} v_5x_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be written compactly as

$$Q_1X_1 = C_1 \tag{31}$$

where $Q_1 \in \Re^{(n-4)\times n}$, $X_1 \in \Re^n$ and $C_1 \in \Re^{(n-4)}$.

The matrix of the **control variable** is given as

$$\begin{bmatrix} -w_6 & -w_5 & -w_4 & -w_3 & -w_2 & -w_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w_6 & -w_5 & -w_4 & -w_3 & -w_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -w_6 & -w_5 & -w_4 & -w_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_6 & -w_5 & -w_4 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -w_6 & -w_5 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -w_6 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_2 & -w_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_3 & -w_2 & -w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_4 & -w_3 & -w_2 & -w_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_5 & -w_4 & -w_3 & -w_2 & -w_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -w_6 & -w_5 & -w_4 & -w_3 & -w_2 & -w_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \vdots \\ u_{n-5} \\ u_{n-4} \\ u_{n-3} \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be written compactly as

$$Q_2U_1 = C_2 \tag{32}$$

where $Q_2 \in \Re^{(n-4)\times(n+1)}$, $U_1 \in \Re^{(n+1)}$ and $C_2 \in \Re^{(n-4)}$.

The augmentation of the matrix of the state variable with the matrix of the control variable gives

$$QW = C \tag{33}$$

where

$$Q = [Q_1 \mid Q_2] \in \Re^{(n-4)\times(2n+1)}$$
$$W = [X_1 \mid U_1] \in \Re^{(2n+1)}$$
$$C = [C_1 \mid C_2] \in \Re^{(n-4)}$$

Q is a coefficient matrix expressed compactly as

$$Q = [q_{ij}] = \begin{cases} 1, & 1 \le i \le n-4, \quad j = i+4; \\ -v_1, & 1 \le i \le n-4, \quad j = i+3; \\ -v_2, & 1 \le i \le n-4, \quad j = i+2; \\ -v_3, & 1 \le i \le n-4, \quad j = i+1; \\ -v_4, & 1 \le i \le n-4, \quad j = i; \\ -v_5, & 2 \le i \le n-4, \quad j = i-1; \\ -w_1, & 1 \le i \le n-4, \quad j = n+i+5; \\ -w_2, & 1 \le i \le n-4, \quad j = n+i+4; \\ -w_3, & 1 \le i \le n-4, \quad j = n+i+3; \\ -w_4, & 1 \le i \le n-4, \quad j = n+i+2; \\ -w_5, & 2 \le i \le n-4, \quad j = n+i+1; \\ -w_6, & 2 \le i \le n-4, \quad j = n+i; \\ 0, & \text{elsewhere.}, \end{cases}$$

C. The Discretized Form

From the discretized objective functional and constraint, the discretized quadratic optimal control problem becomes:

$$\begin{aligned}\text{Minimize} \quad J(x,u) = \frac{M}{2}\Big[x_0^2 + 3x_1^2 + 3x_2^2 + 2x_3^2 + 3x_4^2 \\ + 3x_5^2 + 2x_6^2 + \dots + 2x_{n-3}^2 + 3x_{n-2}^2 \\ + 3x_{n-1}^2 + x_n^2\Big] + \frac{N}{2}\Big[u_0^2 + 3u_1^2 + \\ 3u_2^2 + 2u_3^2 + 3u_4^2 + 3u_5^2 + 2u_6^2 + \dots + \\ 2u_{n-3}^2 + 3u_{n-2}^2 + 3u_{n-1}^2 + u_n^2\Big]\end{aligned}$$

$$\begin{aligned}\text{Subject to} \quad x_{i+5} = v_1 x_{i+4} + v_2 x_{i+3} + v_3 x_{i+2} + v_4 x_{i+1} \\ + v_5 x_i + w_1 u_{i+5} + w_2 u_{i+4} + w_3 u_{i+3} \\ + w_4 u_{i+2} + w_5 u_{i+1} + w_6 u_i \end{aligned} \quad (34)$$

The parametric form of the discretized quadratic optimal control problem becomes

$$\text{Minimize} \quad J(W) = \frac{1}{2} W^T P W + F \quad (35)$$

$$\text{Subject to} \quad QW = C \quad (36)$$

D. The Unconstrained Formulation of the Discretized Form

The discretized unconstrained formulation of the optimal control problem is obtained by applying the Quadratic Penalty Function Method to the parametric form as follows:

$$L(W,\mu) = \frac{1}{2} W^T P W + F + \mu ||QW - C||^2 \quad (37)$$

Expanding the second term of the right hand-side, and grouping like terms, we get

Expanding the second term of the right hand-side, and grouping like terms, we get

$$\begin{aligned} L(W,\mu) &= \frac{1}{2} W^T P W + F + \mu(QW - C)^T \cdot \\ &\quad (QW - C) \qquad (38) \\ &= \frac{1}{2} W^T P W + F + \mu(W^T Q^T - C^T) \cdot \\ &\quad (QW - C) \\ &= \frac{1}{2} W^T P W + F + \mu(W^T W Q^T Q - 2QWC^T \\ &\quad + C^T C) \\ &= \Big(\frac{1}{2} W^T P W + \mu W^T W Q^T Q\Big) - 2\mu C^T Q W \\ &\quad + (F + \mu C^T C) \\ &= \frac{1}{2} W^T (P + 2\mu Q^T Q) W - 2\mu C^T Q W + \\ &\quad (F + \mu C^T C) \\ L(W,\mu) &= \frac{1}{2} W^T P_* W + G^T W + V \qquad (39) \end{aligned}$$

where

$$P_* = P + 2\mu Q^T Q \ \in \Re^{(2n+1)\times(2n+1)} \quad (40)$$

$$G^T = -2\mu C^T Q \ \in \Re^{(2n+1)} \quad (41)$$

$$V = F + \mu C^T C \ \in \Re \quad (42)$$

and $L(W,\mu)$ is the penalized Lagrangian.

The resulting unconstrained formulation (39) is a quadratic programming problem which can be solved by several optimization gradient methods to obtain the numerical solution. The Conjugate Gradient Method (CGM) is adopted in this case to solve (39).

We note that the unconstrained problem (37) approximates the constrained problem (4), so as the penalty parameter, μ in (37) increases, the solution of the unconstrained problem converges to a solution of the constrained problem.

E. The Conjugate Gradient Algorithm for Discretized Optimal Control Problems

The outline of the Conjugate Gradient Algorithm given by [20] is incorporated in writing the code as follows:

Step 0 : **Input** $P,\ Q,\ C,\ F,\ \mu,\ Tol$.

Step 1 : **Initialize** W_0

Step 2 : **Compute**

$P_* = P + 2\mu Q^T Q$

$G = -2\mu C^T Q$

$V = F + \mu C^T C$

Step 3 : $g_0 = P_* W_0 + G$

Step 4 : $P_0 = -g_0$

Step 5 : $\alpha_i = \frac{||g_i||_2^2}{||P_i^T P_* P_i||_2^2}, i = 0, 1, 2, \cdots$

Step 6 : $W_{i+1} = W_i + \alpha_i P_i$

Step 7 : $g_{i+1} = g_i + \alpha_i P_* P_i$

Step 8 : For $i > 1$

if $||g_{i+1}|| \le Tol$, **Stop** , otherwise go to **Step 9**

For $i = 1$

if $||g_{i+1}|| = 0$, **Stop** , otherwise go to **Step 9**

Step 9 : **Compute** $\beta_i = \frac{||g_{i+1}||_2^2}{||g_i||_2^2}$

Step 10 : $P_{i+1} = -g_{i+1} + \beta_i P_i$

Step 11 : **Repeat** Steps 5 to 10.

In our case, the penalty parameter is taken to be $\mu = 1 \times 10^{-3}$ and the tolerance, $Tol = 1 \times 10^{-4}$.

III. RESULTS

Example 1.

$$\text{Minimize } J(x,u) = \int_0^1 (x^2(t) + u^2(t))dt$$

$$\text{Subject to } \dot{x}(t) = 2x(t) + 5u(t)$$

$$x(0) = 1, \quad t \in [0,1] \quad (43)$$

Solution. We note that $a = 1,\ b = 1,\ c = 2,\ d = 5$ and $x_0 = 1$. The analytical objective function value is 0.2954.

TABLE I

Numerical Solutions of the State Variable $x(t)$, Control Variable $u(t)$ and the Objective Functional Value $J(x,u)$

Iterations	Numerical Solution State Variable $x(t)$	Numerical Solution Control Variable $u(t)$	Objective Function Value $J(x,u)$
1	1.1459	1.6335	0.4296
2	0.7600	0.8635	0.3337
3	0.4124	0.3734	0.3037
4	0.2033	0.2212	0.2993
5	0.1352	0.0726	0.2975
6	0.1138	0.0356	0.2975
7	0.1127	0.0348	0.2975
8	0.1123	0.0343	0.2975

Table I shows the comparison of the solutions from the proposed algorithm and the convergence of the objective functional value. The numerical solution of 0.2975 from the proposed algorithm agrees favourably with the analytical solution 0.2954.

Figure (1) shows how fast the state variable decreases and remains stable afterwards. The control variable in figure (2) also shows a downward trend with time. Figure (3) presents a comparison of the objective functional value from the proposed algorithm to that of the analytical solution. It can be seen that the proposed algorithm achieves optimality faster.

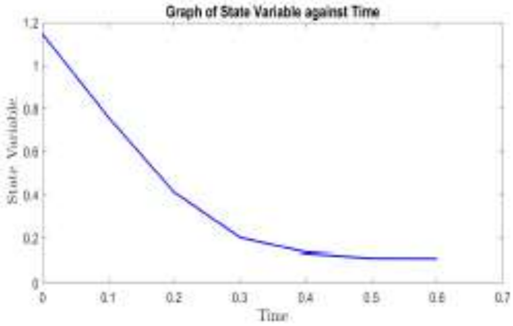

Fig. 1. Graph of State Variable $x(t)$ against Time t

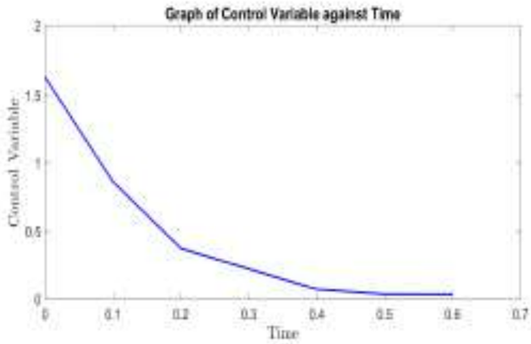

Fig. 2. Graph of Control Variable $u(t)$ against Time t

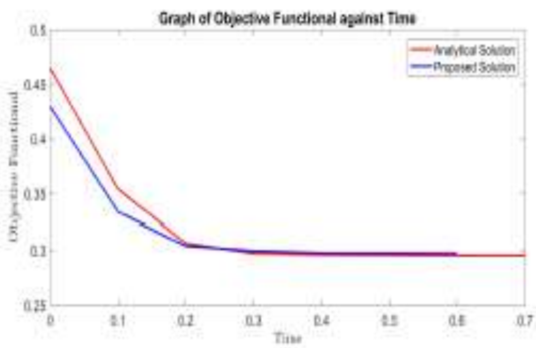

Fig. 3. Graph of Objective Functional $J(x, u)$ against Time t

Example 2.

$$\begin{aligned} \text{Minimize } J(x,u) = & \quad \frac{1}{2}\int_0^1 (2x^2(t) + u^2(t))dt \\ \text{Subject to } \dot{x}(t) = & \quad \frac{1}{2}x(t) + u(t) \\ & \quad x(0) = 1, \quad t \in [0,1] \qquad (44) \end{aligned}$$

Solution. We note that $a = 1$, $b = 0.5$, $c = 0.5$, $d = 1$ and $x_0 = 1$. The analytical objective function value is 0.8642.

TABLE II
Numerical Solutions of the State Variable $x(t)$, Control Variable $u(t)$ and the Objective Functional Value $J(x, u)$

Iterations	Numerical Solution State Variable $x(t)$	Numerical Solution Control Variable $u(t)$	Objective Function Value $J(x, u)$
1	1.0914	2.5093	1.0308
2	0.7367	1.4662	0.9226
3	0.2641	0.8080	0.8734
4	0.1195	0.3182	0.8646
5	0.1180	0.2729	0.8634
6	0.1165	0.1766	0.8632
7	0.1128	0.0207	0.8628
8	0.1097	0.0187	0.8628
9	0.1089	0.0140	0.8628

Table II shows the comparison of the solutions from the proposed algorithm and the convergence of the objective functional value. The numerical solution of 0.8628 from the proposed algorithm compares favourably with the analytical solution 0.8642.

Figure (4) shows how fast the state variable decreases and achieves stability afterwards. The control variable in figure (5) also shows a downward trend with time. Figure (6) presents a comparison of the objective functional value from the proposed scheme to that of the analytical solution and the proposed algorithm is seen to achieve optimality faster.

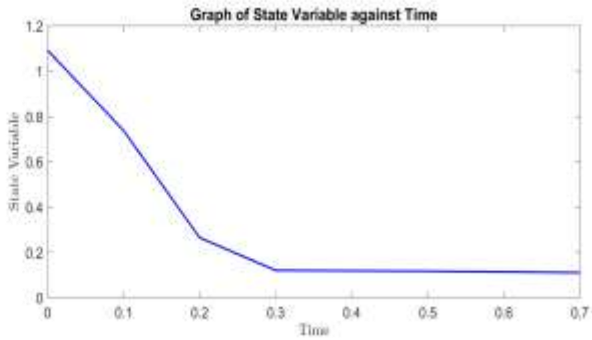

Fig. 4. Graph of State Variable x(t) against Time t

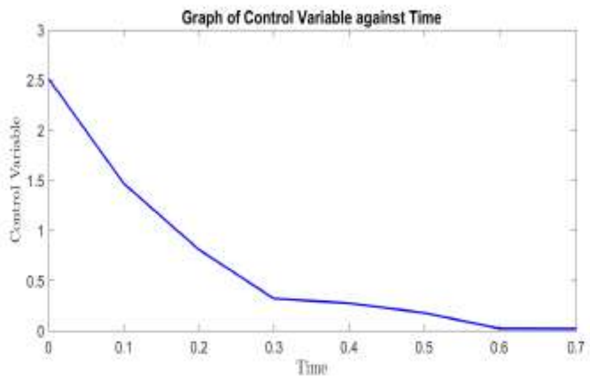

Fig. 5. Graph of Control Variable $u(t)$ against Time t

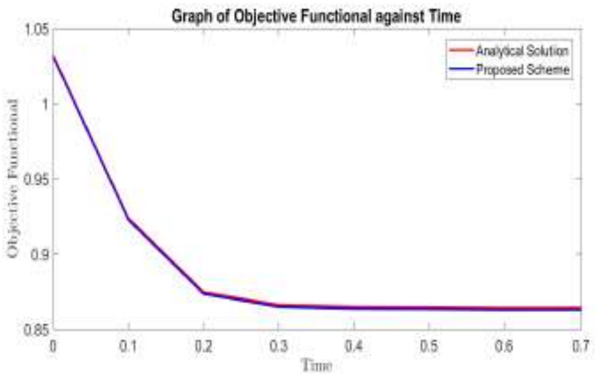

Fig. 6. Graph of Objective Functional $J(x, u)$ against Time t

IV. CONCLUSIONS

As optimal control problems are becoming too complex toallow analytical solutions, a higher-order discretized algorithmfor solving quadratic optimal control problem constrainedby ordinary differential equations is presented. The resultsobtained by the new algorithm after few iterations comparesfavorably with the analytical solution. This establishes thatthe higher discretization schemes gives rise to better accuracyand faster rate of convergence of the solutions. The use of theConjugate gradient method for solving constrained quadraticprogramming problem is also well suited for solving optimalcontrol problems.

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