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Numerical Methods and Convolutional Neural Networks for 3D Medical Image Segmentation

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ABSTRACT

We propose a hybrid framework that integrates classical numerical-PDE based regularization with modern 3D convolutional neural network (CNN) architectures to improve volumetric medical image segmentation. The method augments a 3D U-Net / UNet-style backbone with a learnable PDE layer inspired by PDE-Net and continuous-depth models, and enforces numerical regularity (e.g., diffusion/total variation priors implemented via constrained convolutional kernels) during training. This hybridization targets improved boundary preservation, robustness to noisy or limited annotations, and better generalization across scanners and modalities.

Keywords:- 3D medical image segmentation, numerical methods, PDE-Net, 3D U-Net, convolutional neural networks, neural ODEs, regularization.

I. INTRODUCTION

3D volume segmentation networks: 3D U-Net extended the 2D U-Net for volumetric segmentation and remains a baseline for many tasks. [1] V-Net introduced a fully convolutional volumetric architecture with a dice-loss training strategy for medical volumetric tasks. [2] More recent architectures combine convolution and attention or transformer blocks for improved global context (e.g., Swin Unet3D). [5]

Numerical / PDE approaches: PDE and variational methods (diffusion, level sets, TV) have a long history in image segmentation and denoising and offer explicit control over smoothness and boundaries. Operator splitting and phase-field models remain active areas for numerical image segmentation. [8]

Bridging PDEs and deep learning: PDE-Net shows how convolutional filters can be constrained to learn differential operators and discover PDE structure from data. Neural ODEs provide a continuous-depth perspective useful for stability and for interpreting residual networks as numerical integrators of ODEs. These ideas suggest routes to embed numerical operators into trainable networks. [3]

Recent practical advances include efficient 3D segmentation variants (e.g., ES-UNet) and frameworks that aim to generalize under small training data regimes. These works motivate integrating mathematical priors to improve robustness. [5]

II. PROPOSED METHOD

A. Overview

We propose PDE-Regularized 3D U-Net (PDE3D-UNet): a volumetric segmentation network that interleaves standard 3D convolutional encoder–decoder blocks with *PDE layers* that (a) perform constrained convolutional operations representing learned finite-difference approximations of differential operators (gradient, Laplacian, anisotropic diffusion), and (b) optionally run small, stable numerical updates (implicit/Crank–Nicolson / operator-split steps) to impose smoothness priors on intermediate feature maps.

Schematic: Input volume → Encoder blocks → PDE layer(s) → Bottleneck → Decoder blocks + skip connections → Output segmentation. The PDE modules are differentiable and trained end-to-end.

B. Mathematical formulation

Let $I(x)$be the input image volume and $F_\theta(I)$the CNN feature maps parameterized by θ. We introduce a PDE layer that evolves features $u(x,t)$according to a learnable evolution law:

$$\partial_t u(x,t) = \mathcal{D}_\phi[u](x,t) + \mathcal{N}_\psi(u,x,t),$$

where $\mathcal{D}_\phi$is a (linear or nonlinear) differential operator parameterized by constrained convolutional kernels ϕ(representing finite-difference approximations of ∇, Δ, anisotropic diffusion coefficients), and $\mathcal{N}_\psi$is a learned local nonlinear response (small MLP or 1×1×1 conv) parameterized by ψ. This is in the spirit of PDE-Net and continuous models (Neural ODE). [3]

C. Discretization and stable solver choices

We discretize time with a small number $K$of steps (e.g., 1–5) inside each PDE layer. For stability and computational efficiency:

• Use semi-implicit or Crank–Nicolson style updates for diffusion terms (to allow larger time steps without instability).
• Constrain convolutional kernels so their moment sums correspond to derivative approximations (e.g., Laplacian stencil) — following PDE-Net constraints — enabling interpretability and numerical consistency.

D. Loss function

Total loss

$$\mathcal{L} = \mathcal{L}_{\text{seg}} + \lambda_{\text{PDE}} \mathcal{L}_{\text{reg}} + \lambda_{\text{aux}} \mathcal{L}_{\text{aux}}.$$

• $\mathcal{L}_{\text{seg}}$— standard segmentation loss (Dice + cross-entropy).
• $\mathcal{L}_{\text{reg}}$— numerical regularization enforcing PDE-consistency (e.g., penalties on divergence of flux or deviations from implicit PDE step) and optional TV penalty on the predicted mask to promote boundary sharpness.
• $\mathcal{L}_{\text{aux}}$— auxiliary losses (deep supervision at decoder levels).

Hyperparameters $\lambda_\cdot$ tuned by validation.

E. Implementation details

• Backbone: 3D U-Net (encoder/decoder stages with 3×3×3 convolutions, instance norm, ReLU). [1]
• PDE layer: 3D convolutions with constrained kernels (3×3×3 or 5×5×5), time stepping $K = 2$by default, semi-implicit solver implemented with one Jacobi/Gauss-Seidel inner iterate per step for efficiency.
• Training: AdamW optimizer, learning rate schedule with cosine annealing, data augmentation (random crop, flip, intensity shifts).
• Hardware and memory strategies: mixed precision, patch training (if full volumes don't fit GPU). See ES-UNet for efficient 3D strategies. [6]

III. EXPERIMENTAL PLAN

A. Datasets

We plan to evaluate on public benchmarks widely used for 3D segmentation:

• **BraTS** (brain tumor segmentation) — multi-modal MRI volumes with expert tumor labels.
• **LiTS** (liver and tumor segmentation) — CT volumes. (Use training/validation/test splits standard to each challenge).

B. Baselines and ablations

Baselines: 3D U-Net, V-Net, Swin Unet3D (hybrid transformer), ES-UNet (efficient 3D UNet). Ablations: (1) backbone only, (2) backbone + PDE layer(s) as regularizer (loss term only), (3) backbone + embedded PDE layers (our full model), (4) varying number of PDE steps and different solver choices. [10]

C. Metrics

Dice similarity coefficient (DSC), Hausdorff distance (95th percentile), precision/recall on structures, and robustness measures across noisy/low-label regimes.

D. Hypotheses / Expected outcomes

We hypothesize: (1) PDE3D-UNet improves boundary accuracy (lower Hausdorff) and is more robust when training labels are limited; (2) semi-implicit PDE layers allow stable enforcement of diffusion-like priors without harming gradient flow owing to differentiability; (3) constrained kernels give interpretability and sometimes faster convergence. These hypotheses are motivated by earlier works that show benefits of both PDE priors and specialized 3D architectures. [3] [8]

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IV. DISSCUSSION .

The hybrid approach aims to combine explicit mathematical priors (which excel at smoothness and sharp boundary control) with the expressivity of modern 3D deep nets. Potential limitations include additional computational cost and careful tuning of PDE hyperparameters. Integrating PDE layers tightly with GPU-friendly solvers and exploring sparse updates or learned time-step control (via Neural ODE mechanisms) are promising directions for scalability. [2]

V. CONCULSION AND FUTURE WORK

We presented a clear blueprint for integrating numerical PDE methods with 3D CNN segmentation networks. The next step is empirical: implement PDE3D-UNet, run the planned experiments on BraTS and LiTS, and compare quantitatively with baselines. Future extensions include learned anisotropic diffusion with anatomical priors, multi-modal fusion, and deploying lightweight variants for clinical hardware constraints.

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[10] Additional resources and code repositories for the cited architectures (SwinUnet3D, ES-UNet) and PDE-Net implementations referenced in the text.