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Versatile of Hybrid Numerical Asymptotic Boundary Element Method framework

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ABSTRACT

In this paper we tend to summarise abundant of the one scattering HNA work before this paper, within a general framework which is able to conjointly embrace a broader category of issues during this paper. we tend to conjointly introduce and analyse a replacement approximation house, as an alternate to what has been used antecedently. before this thesis, HNA ways have solely been designed for issues of plane wave incidence. Intuitively, this approximation could also be taken as double the traditional spinoff on the edges which will see the incident waves. At sufficiently high frequencies, this might be an acceptable approximation, but it's not controllably correct. The Hybrid Numerical straight line technique improves on the Physical Optics Approximation by approximating the diffracted waves numerically.In previous methods such as Ψ can be written explicitly in terms of u^{inc}, although an approximate representation may be sufficient. This is the case for the penetrable obstacles of in which a beam tracing algorithm is used to approximate reflections inside Ω_Γ.

Keywords:- HNA

I. REPRESENTATION ON A SINGLE SIDE

In related literature, there appears to be no single consistent definition of the term polygon, so we shall clarify a definition that is appropriate for what follows.

DEFINITION 1.1

(Polygon)**.** We say Ω_- is a polygon if it is a bounded Lipschitz open set with a boundary $\partial\Omega$ consisting only of straight line segments, such that the endpoint of every segment is connected to one other endpoint of another segment.

We note that Definition 2.1 permits multiple disconnected shapes, whereas other conventions in the literature do not. Many results that follow hold for a subclass of polygons, which we define now (as in e.g., [50, Definition 1.1])

DEFINITION 1.2

(Non-trapping polygon)**.** We say that a polygon Ω_- (in the sense of Definition 2.1) is non-trapping if:

(i) No three vertices of $\partial\Omega$ are co-linear, i.e. they lie in a straight line.

(ii) For a ball B_R with radius $R > 0$ sufficiently large that $\Omega_- \subset B_R$, there exists a $T(R) < \infty$ such that all billiard trajectories that start inside of $B_R \setminus \Omega_-$ that start at time zero $T = 0$ and miss the vertices of $\partial\Omega$ will leave B_R by time $T(R)$.

In this chapter we assume that scattering obstacle Ω_- of the general problem statement of §1.1 is a convex polygon, which we denote Ω_Γ with boundary $\partial\Omega = \Gamma$. We

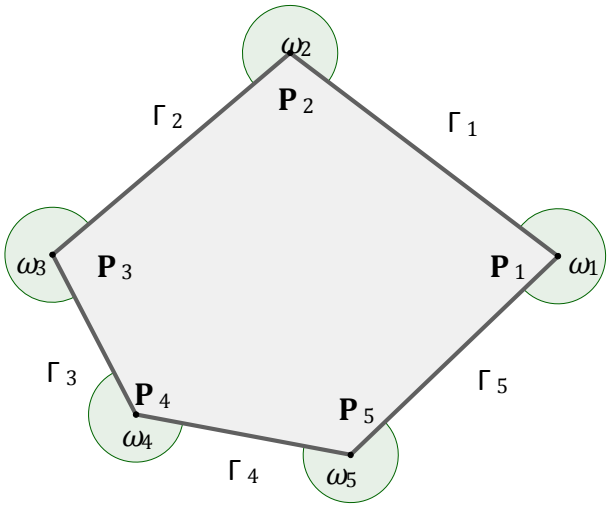


Figure 1.1: Example of a five-sided polygon ($n_\Gamma = 5$) with boundary Γ.

now define a range of parameters related to the geometry of Ω_Γ, on which subsequent bounds will depend. The parameters in these definitions will hold for any convex polygon in this thesis.

DEFINITION 1.3

(Parameters of a convex polygon)**.** For an n_Γ-sided polygon (in the sense of Definition 2.1) with boundary Γ, we denote by $\mathbf{P}_j$ and $\mathbf{P}_{j+1}$ the nodes at the endpoints of each side Γ_j, for j = 1,...,n_Γ setting $\mathbf{P}_{n\Gamma+1} := \mathbf{P}_1$. Conventionally, the vertices are indexed anti-clockwise. We denote by $L_j := |\mathbf{P}_{j+1} - \mathbf{P}_j|$ the length of Γ_j, $\tilde{L}_j := \sum_{\ell=1}^{j} L_\ell$, and $L_\Gamma := \tilde{L}_{n\Gamma}$. The jth exterior angle is denoted ω_j, hence for a convex polygon we have $\omega_j \in (\pi, 2\pi)$. Finally we choose the constant $c_* > 0$ such that $kL_j \geq c_*$ for j = 1,...,n_Γ (e.g. $c_* = \min_j\{kL_j\}$), and $L_* := \max_j\{L_j\}$

$$\Gamma(s) = \mathbf{P}_j + \frac{s - \tilde{L}_{j-1}}{L_j}(\mathbf{P}_{j+1} - \mathbf{P}_j), \quad s \in [\tilde{L}_{j-1}, \tilde{L}_j), \quad j = 1, \ldots, n_\Gamma$$

Figure 2.1 depicts a five-sided polygon with certain parameters of Definition 1.3.

We parametrise Γ by

$$\mathbf{x}(1.1)$$

and $\Gamma^-_j \cup \Gamma_j \cup \Gamma^+_j$, which is the straight line containing Γ_j, extended infinitely in both directions (defined in more detail in Chapter 3), by

$$\mathbf{y}_j(s) = \mathbf{P}_j + \frac{s - \tilde{L}_{j-1}}{L_j}(\mathbf{P}_{j+1} - \mathbf{P}_j), \quad s \in \mathbb{R}, \quad j = 1, \ldots, n_\Gamma$$

.

Now we consider the scattering problem , with solution u, for the case where the scattering object Ω_- is a convex polygon. In such a case, there are two key physical components of the scattered field u^s. Firstly, the waves reflected by the edges Γ_j, for j = 1,...,n_Γ. As we will see, these can be written explicitly without the need for numerical approximation. The second type of wave which contributes to the scattered field are the diffracted waves emanating from the corners $\mathbf{P}_j$, for j = 1,...,n_Γ. Recall that with a BIE formulation , we are instead solving for

$\partial^+_{\mathbf{n}} u$. In such a case, the diffracted waves move in just two directions along each side Γ_j, for j = 1,...,n_Γ, making this formulation an ideal approach for approximating the diffracted waves. Moreover, it is possible to separate explicitly the oscillatory behaviour of these two diffracted terms; the remaining (unknown) term which must be approximated numerically is non-oscillatory . The full mathematical derivation depends on the incident wave u^i (see [16, §3] for plane waves, see Chapter 3 of this thesis for more general incidence), however the HNA ansatz for a single impenetrable convex polygon may be written as:

$$(\mathbf{x}_\Gamma(s)) = \overbrace{\Psi(\mathbf{x}_\Gamma(s))}^{} + \underbrace{v_j^+(\mathbf{x}_\Gamma(s - \tilde{L}_{j-1}))}\, e^{iks} + \underbrace{v_j^-(\mathbf{x}_\Gamma(\tilde{L}_j - s))}\, e^{-iks},$$

Reflected terms Diffracted

___ terms ∂u

∂**n**

First unknown envelope Second unknown envelope

where s ∈ [L˜$_{j-1}$,L˜$_j$], for j = 1,...,n$_\Gamma$, with $\mu(z) := e^{-iz}H_1^{(1)}(z)/z$, the (unknown) amplitudes of the diffracted waves are represented by

$$v_j^+(s) := \frac{ik^2}{2}\int_0^\infty \mu\big(k(s+t)\big)e^{ik(t-\widetilde{L}_{j-1})}u\big(\mathbf{y}_j(\widetilde{L}_{j-1}-t)\big)\,dt, \qquad s\in[0,L_j],$$

$$v_j^-(s) := \frac{ik^2}{2}\int_0^\infty \mu\big(k(s+t)\big)e^{ik(\widetilde{L}_j+t)}u\big(\mathbf{y}_j(\widetilde{L}_j+t)\big)\,dt, \qquad s\in[0,L_j], \qquad (2.3)$$

whilst Ψ represents the leading order asymptotics corresponding physically to the reflected terms, hence we shall see later that Ψ is zero on the sides which are not illuminated by the incident wave. The term Ψ is often referred to as the Physical Optics Approximation for single scatterers, Associate in Nursing approximation that ignores diffracted waves. Intuitively, this approximation could also be taken as doubly the conventional spinoff on the perimeters that may see the incident waves. At sufficiently high frequencies, this might be an appropriate approximation, but it's not controllably correct. The Hybrid Numerical straight line technique improves on the Physical Optics Approximation by approximating the diffracted waves numerically. In previous methods such as [35], Ψ can be written explicitly in terms of u^{inc}, although an approximate representation may be sufficient. This is the case for the penetrable obstacles of [31], in which a beam tracing algorithm is used to approximate reflections inside Ω$_\Gamma$. This takes the form of an infinite series, which must be truncated, and is thus only approximate (although in theory, controllably accurate). However, the representation for the diffracted waves for penetrable Ω$_\Gamma$ is more complex, and does not fit within the framework (2.2). Similarly, the ansatz for non-convex obstacles of [15] contains an additional term, and does not fit within (2.2).

It is the envelopes $v_j^\pm$ that are approximated numerically, indeed our hp approximation will converge exponentially (from [35, Theorem 5.2]) if the following assumption holds.

ASSUMPTION 1.4. There exists a term M(u) such that:

(i) The functions $v_j^\pm$, *for* $j = 1,\dots,n_\Gamma$, are analytic in the right half-plane Re[s] > 0, where they satisfy the bounds

$$|v_j^\pm(s)| \le \begin{cases} C_j^\pm M(u)k|ks|^{-\delta_j^\pm}, & 0<|s|\le 1/k,\\ C_j^\pm M(u)k|ks|^{-1/2}, & |s|>1/k,\end{cases}$$

where δ$_j^+$,δ$_j^-$ ∈ (0,1/2) are given by δ$_j^+$:= 1 − π/ω$_j$ and δ$_j^-$:= 1 − π/ω$_{j+1}$, C$_j^+$ depends only on c$_*$ and ω$_j$, C$_j^-$ depends only on c$_*$ and ω$_{j+1}$

(ii) M(u) depends on the size of the solution u to (1.4) − (1.6) grows at most algebraically with k, i.e. there exists a β ≥ 0 such that

$$M(u) \,.\, k^\beta, \qquad \text{for all } k \ge 0, \qquad (1.4)$$

where a . b means that a ≤ cb, where the constant c depends only on the geometry of Ω$_-$.

To summarise, in order to design HNA methods we must be able to do the following:

(i) Represent leading order behaviour accurately via Ψ.

(ii) Show that Assumption 2.4 holds.

Prior to this thesis, HNA methods have only been designed for problems of plane wave incidence, which are defined as:

$$u^{inc}(\mathbf{x}) = u^{inc}_{PW}(\mathbf{x};\alpha) := e^{ik\mathbf{d}_\alpha\cdot\mathbf{x}}, \qquad \text{where } \mathbf{d}_\alpha := (\cos\alpha,-\sin\alpha). \qquad (1.5)$$

Unlike previous HNA papers (for e.g. [16, §2]), we choose α to be the angle that the plane wave is emanating from, measured against the x$_1$-axis. This is to be consistent with previous literature on Embedding Formulae.

REMARK 1.5 (The constant M(u) for plane waves)**.** For problems of plane wave incidence (as in (2.5)) scattering by a single convex polygon Ω$_-$, Assumption 2.4 has been shown to hold with

M(u) = M$_\infty$(u) := sup |u(**x**)|,

x∈Ω+

[35, Theorem 3.2]. Numerical experiments of [16] and [15] suggest that for problems of plane wave incidence with convex Ω_-, we have $M_\infty(u) \sim 1$, although the strongest theoretical bound is currently $M_\infty(u) \cdot k^{1/2} \log^{1/2} k$, for $k \geq k_0$ where k_0 is a constant independent of k, (see [35, Theorem 4.3] for star shaped polygons, this is generalised to non-trapping polygons in Corollary 4.7 of this thesis). In §3.2.1 we will consider cases for which Assumption 2.4(i) holds, but with $M_\infty(u)$ unbounded, hence requiring a different choice of M(u).

The framework we present does not include the non-convex work of [15]; this requires an extra term in the ansatz which captures the oscillatory nature of the waves diffracted by the corners, as they are reflected by the non-convex sides.

1.2 Approximation space

We now design an approximation space to represent efficiently the diffracted waves emanating from the corners of the convex polygon Γ,

$$v_\Gamma(s) := \frac{1}{k}\left(v_j^+(s-\widetilde{L}_{j-1})\mathrm{e}^{\mathrm{i}ks} + v_j^-(\widetilde{L}_j - s)\mathrm{e}^{-\mathrm{i}ks}\right), \quad s \in \left[\widetilde{L}_{j-1}, \widetilde{L}_j\right], \; j = 1, \ldots, N_\Gamma, \tag{2.6}$$

a term containing all of the unknown components of (2.2) (recall that the reflected waves can be represented explicitly). The scaling by 1/k ensures that v_Γ is dimensionless. Instead of approximating v_Γ by piecewise polynomials, we use our knowledge of wise polynomials.

Both of these are singular as s tends to zero, thus the polynomiale e the oscillations $e^{\pm iks}$ and approximate both $v_j^+(s - L_{j-1})$ and $v_j^-(L_{j-1} - s)$ by piece-space needs a stratified mesh to confirm a robust approximation. during this section we have a tendency to gift 2 such approximation areas, each stratified and enriched with oscillating basis components. 1st we have a tendency to gift the overlapping-mesh house, that has been employed in all previous HNA ways, enclosed here for completeness and for a simple comparison against the new mesh.

This house consists of 2 overlapping meshes, stratified towards opposite corners. second we have a tendency to propose a brand new different approach, the single-mesh space; actuated by doubtless easier implementation.

We shall see in Chapter four that the approximation areas we have a tendency to gift here aren't solely acceptable for a brand new category of incident fields, however may be used on a polygonal shape within a multiple scattering configuration.

The overlapping-mesh hybrid space

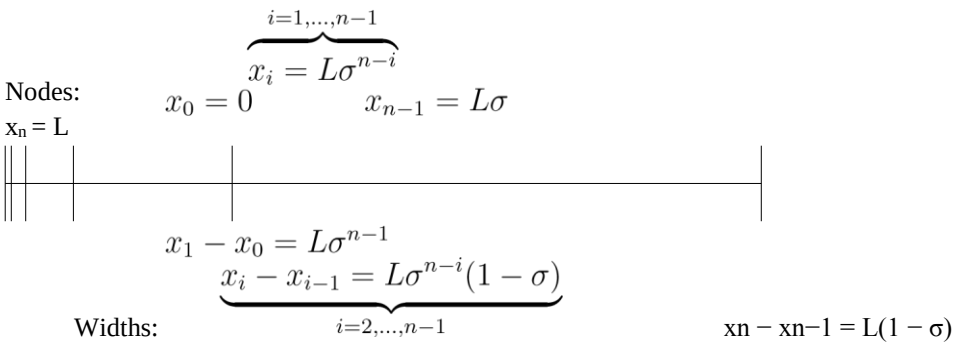
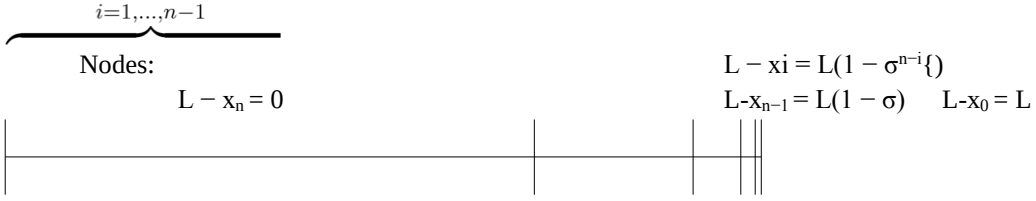


Figure 2.2: The nodes and widths of the mesh as described in Definition 2.6



$$x_n - x_{n-1} = L(1-\sigma) \qquad x_1 - x_0 = L\sigma^{n-1}$$
$$\underbrace{x_i - x_{i-1} = L\sigma^{n-i}(1-\sigma)}_{i=2,\ldots,n-1}$$

Widths:

Figure 2.3: The nodes and widths of the mesh as described in Definition 2.6 subtracted from L, to construct a mesh graded in the opposite direction.

DEFINITION 1.6.

Given L > 0, n ∈ N and a grading parameter σ ∈ (0,1), we denote by $\mathcal{M}_n^<(0,L) = \{x_0,\ldots,x_n\}$ the geometrically graded mesh on [0,L] with n layers graded towards 0, whose n + 1 meshpoints x_i are defined by

$x_0 := 0, \qquad x_i := L\sigma^{n-i}, \qquad$ for i = 1,...,n.

For a vector **p** = $(p_1,\ldots,p_n) \in (N_0)^n$ we denote by $\mathcal{P}^<_{\mathbf{p},n}$ the space of piecewise polynomials on $\mathcal{M}_n^<(0,L)$ with degree vector **p**, i.e.

$$\mathcal{P}^<_{\mathbf{p},n}(0,L) := \left\{\rho \in L^2(0,L) : \rho|_{(x_{i-1},x_i)}\right.$$

is a polynomial of degree at most p_i for i = 1,...,no.

Where p_i = p for i = 1,...,n, we write $\mathcal{P}^<_{p,n}$ *for* $\mathcal{P}^<_{\mathbf{p},n}$.

The space $\mathcal{P}^<_{\mathbf{p},n}(0,L_j)$ is designed to approximate v_j^+, enrichment with oscillatory functions provides a space suitable to approximate $v_j^+ e^{\pm iks}$. We first define two spaces for each side Γ_j, j = 1,...,n_Γ, using $n_j \in N$ to determine the degree of mesh grading and the vectors $\mathbf{p}_j$ to determine the polynomial degree on each mesh element:

$$\overline{\overline{V}}_j^+ := \left\{v \in L^2(0,L_\Gamma) : v|_{(\widetilde{L}_{j-1},\widetilde{L}_j)}(s) = \widetilde{\rho}(s-\widetilde{L}_{j-1})e^{iks}, \quad \widetilde{\rho} \in \mathcal{P}^<_{\mathbf{p}_j,n_j}(0,L_j)\right.$$
$$\left.\rho|_{(0,L_\Gamma)\setminus(\widetilde{L}_{j-1},\widetilde{L}_j)} = 0\right\},$$
$$\overline{\overline{V}}_j^- := \left\{v \in L^2(0,L_\Gamma) : v|_{(\widetilde{L}_{j-1},\widetilde{L}_j)}(s) = \widetilde{\rho}(\widetilde{L}_j - s)e^{-iks}, \quad \widetilde{\rho} \in \mathcal{P}^<_{\mathbf{p}_j,n_j}(0,L_j)\right.$$
$$\left.\rho|_{(0,L_\Gamma)\setminus(\widetilde{L}_{j-1},\widetilde{L}_j)} = 0\right\}.$$

In the space $\overline{\overline{V}}_j^-$, the argument of ˜ρ decreases as s increases. This is related to the mesh depicted in Figure 2.3. The overlapping-mesh approximation space can now be defined as ⁼HNA

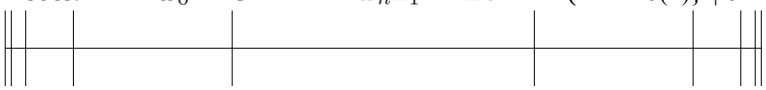
$$V_N(\Gamma) := \text{span},$$

⁼HNA where N refers to the number of degrees of freedom in the space $V_N(\Gamma)$, and depends on parameters $\mathbf{p}_j$ and n_j for j = 1,...,n_Γ. **The single-mesh hybrid space**

$$\left\{\bigcup_{j=1}^{n_\Gamma}(\overline{\overline{V}}_j^+ \cup \overline{\overline{V}}_j^-)\right\}$$

i=n,...,2n−2 $\underbrace{x_i = L\sigma^{n-i}}_{i=1,\ldots,n-1}$

Nodes:xn = $x_0 = 0 \qquad x_{n-1} = L\sigma$ Lxz(1i =-Lσ(1)}⁻| σxi−2nn+1−1{)= L



$$x_1 - x_0 = L\sigma^{n-1} \qquad x_n - x_{n-1} = L - 2\sigma \qquad x_{2n-1} - x_{2n-2} = L\sigma^{n-1}$$
$$\underbrace{x_i - x_{i-1} = L\sigma^{n-i}(1-\sigma)}_{i=2,\ldots,n-1} \qquad \underbrace{x_i - x_{i-1} = L\sigma^{i-n}(1-\sigma)}_{i=n+1,\ldots,2n-2}$$

Widths:

DEFINITION 1.7.

Given L > 0, n ∈ N and a grading parameter σ ∈ (0,1/2), we denote by Mn(0,L) = $\{x_0,\ldots,x_{2n-1}\}$ the symmetric geometrically graded mesh on [0,L] with n layers in each direction, whose 2n meshpoints x_i are defined by

$x_0 := 0$, $x_i := L\sigma^{n-i}$, for, $x_i := L(1-\sigma^i)$, for i = n,...,2n − 2, x_{2n-1} $\quad i = 1,\ldots,n-1 \quad := L.$

For a vector $\mathbf{p} = (p_1,...,p_n) \in (N_0)^n$ we denote by $P_{\mathbf{p},n}$ the space of piecewise polynomials on Mn(0,L) with degree vector **p**, i.e.

$$\mathcal{P} \quad , L \quad \left\{ \rho \in L^2(0,L) : \rho|_{(x_{i-1},x_i)} \qquad \rho|_{(x_{2n-1-i},x_{2n-i})} \right\}$$

and $\mathbf{p}_{,n}(0$) :=.

are polynomials of degree at most p_i for i = 1,...,n

In the case where $p_i = p$ for i = 1,...,n, we write Pp,n for $P_{\mathbf{p},n}$.

We first define two spaces for each side Γ_j, j = 1,...,n_Γ, using $n_j \in N$ to determine the degree of mesh grading and the vectors $\mathbf{p}_j$ to determine the polynomial degree on each mesh element:

$$\overline{V}_j^+ := \Big\{ v \in L^2(0,L_\Gamma) : v|_{(\tilde{L}_{j-1},\tilde{L}_j)}(s) = \tilde{\rho}(s-\tilde{L}_{j-1})\mathrm{e}^{\mathrm{i}ks}, \tilde{\rho} \in \mathcal{P}_{\mathbf{p}_j,n_j}(0,L_j)$$

$$\rho|_{(0,L_\Gamma)\setminus(\tilde{L}_{j-1},\tilde{L}_j)} = 0 \Big\},$$

$$\overline{V}_j^- := \Big\{ v \in L^2(0,L_\Gamma) : v|_{(\tilde{L}_{j-1},\tilde{L}_j)}(s) = \tilde{\rho}(\tilde{L}_j - s)\mathrm{e}^{-\mathrm{i}ks}, \tilde{\rho} \in \mathcal{P}_{\mathbf{p}_j,n_j}(0,L_j)$$

$$\rho|_{(0,L_\Gamma)\setminus(\tilde{L}_{j-1},\tilde{L}_j)} = 0 \Big\}.$$

As is explained in Remark 2.8, to avoid ill conditioning of the discrete system we must remove certain basis functions.

Vej := span $\left(\left\{ v \in V_j^- : v|_{[\tilde{L}_{j-1},\tilde{L}_{j-1}+x_{\tilde{n}_j}]} = 0 \right\} \cup \left\{ v \in V_j^+ : v|_{[\tilde{L}_j - x_{\tilde{n}_j},\tilde{L}_j]} = 0 \right\} \right)$ where

$$x_{\mathrm{n}e_j} := \max x_i \in \mathrm{Mn_j}(0,L_j) \text{ such that } x_i \le \alpha_j \frac{2\pi}{k} \Big\} \qquad (1.7)$$

and α_j is a parameter chosen such that $0 < \alpha_j < L_j k/(4\pi)$, bounded independently of k and and $\mathbf{p}_j$, used to fine tune the space. Put simply, there are two basis functions on elements sufficiently far from the corners, and one basis element on elements close to the corners. The parameter α_j determines the threshold referring to precisely what is meant by sufficiently close. Hence the single-mesh approximation space is defined as

$n\Gamma$

$V_{\mathrm{N}}\mathrm{HNA}(\Gamma) := \mathrm{span}\ [\ \tilde{V}_j.$

$j=1$

REMARK 1.8

(Why elements of $\overline{V}_j^{\pm}$ are removed)**.** Since the mesh is strongly graded to approximate the singularities of $v_j^{\pm}$, some of its elements are much smaller than the wavelength of the problem, thus $e^{\pm iks}$ are roughly constant on them and the functions $of\ \overline{V}_j^+$ supported on these elements are numerically indistinguishable from those on $\overline{V}_j^-$, leading to an ill-conditioned discrete system of Galerkin methods set in $\overline{V}_j^+ \cup \overline{V}_j^-$. To avoid this, in these elements we maintain only one of these two contributions. Intuitively, α_j can be thought of as the value such that in all elements with distance multiplied with only one of the waves $e^{\pm iks}$. As the parametere α_j increases, fewer from one of the segment endpoints smaller than α_j, the space V_j supports polynomials degrees of freedom are used and the conditioning of the discrete system is improved, but the accuracy of the method is reduced, hence care must be taken when selecting α_j.

As has become standard for HNA BEM , in the numerical experiments of §3.2.3 and §4.5 we choose a grading parameter of $\sigma = 0.15$, which is a prudent over-refinement of the value suggested by [32, Theorem 3.2]. We note that our definition of each approximation space results in symmetric grading and distribution of polynomial degrees. For the more complex asymmetric definition on the overlapping mesh, see for example [35, §5].

We shall shortly present a new result, a best approximation result analogous to the overlapping mesh case of adapted for the single mesh. First, we motivate why a modification to the overlapping mesh theorem is required, outlining the differences between the two spaces. The basis functions on the graded regions of the mesh, which are designed to handle the singularities of $v_j^{\pm}$, will also be used to approximate a smooth wave

propagating in the opposite direction, for which the $2(n_j - ne_j)-1$ elements $[Lej_{-1} + x_n e_j, Lej_{-1} + x_n e_{j+1}], \ldots, e\ [Lej - x_n e_{j+1}, Lej_{-1} + x_n e_j]$ coincide grading is not necessary. The elements of the space V_j supported on the larger central

with those of span($V_j^+ \cup V_j^-$), thus the HNA approximation results of [35, §5] apply. However, on the first and final n_j mesh elements of Mn(0,L_j), the elements of the discrete space contain only one of the two oscillating factors $e^{\pm iks}$. For example, in functions of the form $e^{iks}\rho(s - L_{j-1}e)$ for a polynomiale ρ, need to approximate both the the first n_j smaller elements $[\widetilde{L}_{j-1}, L_{j-1} + x_1], \ldots, [L_{j-1} + x_{\tilde{n}_j - 1}, \widetilde{L}_{j-1} + x_{\tilde{n}_j}]$, the basis

singular function $v_j^+(s - L_j)e^{iks}$. Here the approximation theory of [35] applies again, and the smooth function $v_j^-(L_j - s)e^{-iks}$, for which we prove approximation bounds in the next theorem. These correspond to the approximation of $v_j^-(\widetilde{L}_j - s)e^{-2iks}$ away from its singularity with piecewise polynomials. In particular we want to control the dependence of the error on the wavenumber k. In the remaining n_j elements closest to $\widetilde{L}_j$ the same reasoning applies with basis functions $e^{-iks}\rho(\widetilde{L}_j - s)$. of the form

For the overlapping-mesh space, best approximation estimates were derived in [35, Theorem 5.4] (summarised shortly in Corollary 2.11). The single-mesh space is a relatively new approach to HNA methods, and prior to this thesis no analogous result had been derived. The following theorem provides such estimates, and illustrates the dependence of the best approximation on the parameter α_j.

THEOREM 1.9.

Suppose that the polynomial degree p_j is constant across the elements of the side Γ_j, that Assumption 2.4 holds, $n_j \geq c_j p_j$ for $c_j > 0$ of Definition 2.3, and $x_{n_j} \leq L_j k/(2+\varrho_j)\pi$ for some $0 < \varrho_j \leq 1$. Then we have the following bound, concerning the best approximation of the single-mesh space, on a single side Γ_j of a convex polygon:

$$\inf_{w_N \in \widetilde{V}_j} \left\| v_j^{\pm} e^{\pm ik\cdot} - w_N \right\|_{L^2(0,L_j)} \tag{1.8}$$

$$\leq C_j M(u)\sqrt{k}\left((kL_j)^{1/2-\delta_j^{\pm}} + \log^{1/2}(2+kL_j) + \sqrt{k}(kI_j)^{-\delta_j^{\pm}}\right)e^{-p_j\tau_j^{\pm}} \tag{1.9}$$

where

$$\tau_j^{\pm} := \min\left\{ c_j^{\pm} |\log\sigma| (1/2 - \delta_j^{\pm}),\ \log\frac{1+\sigma^{1/2}(2-\sigma)^{1/2}}{1-\sigma},\ \log\left(1 + \epsilon_j + \epsilon_j^{1/2}(2+\epsilon_j)^{1/2}\right) \right\},$$

$I_j := L_j - x_n e_j(1 + \varrho_j/2) > 0$, and

$$C_j := \max\left\{ C_4, \max_{\pm}\{C_j^{\pm}\}\sqrt{x_{\tilde{n}_j}}\frac{2}{\epsilon_j + \epsilon_j^{1/2}(2+\epsilon_j)^{1/2}} e^{2\alpha_j\pi\sqrt{\epsilon_j(\epsilon_j+2)}} \right\},$$

with C_4 as in [35, Theorem 5.5].

Proof. We give only the details for the case of v_j^+, that of v_j^- follows by similar arguments. From the definition

$\widetilde{V}_j$ and $\overline{V}_j^{\pm}$

of $\leq$ we have

$$\inf_{w_N \in \widetilde{V}_j} \left\| v_j^+ e^{ik\cdot} - w_N \right\|^2_{L^2(0,L_j)}$$

$$\inf_{w_N \in \overline{V}_j^+} \left\| v_j^+ e^{ik\cdot} - w_N \right\|^2_{L^2(0, L_j - x_{\tilde{n}_j})} + \inf_{w_N \in \overline{V}_j^-} \left\| v_j^+ e^{ik\cdot} - w_N \right\|^2_{L^2(L_j - x_{\tilde{n}_j}, L_j)}.$$

By Assumption 2.4, $g(z) = v_k^+(z/k)$ satisfies the estimates required in [35, Theorem 5.2], and using $|e^{iks}| = 1$ the first term on the right-hand side is hence bounded as in [35, Theorem 5.4], leading to the first two terms in the brackets in (2.8). Focusing on the second term, we multiply by $e^{ik\cdot}$ and scale by a factor k:

$$\inf_{w_N \in \overline{V}_j^-} \left\| v_j^+ \mathrm{e}^{\mathrm{i}k\cdot} - w_N \right\|_{L^2(L_j - x_{\tilde{n}_j}, L_j)} = \inf_{P \in \mathcal{P}_{p_j,n}(0,L_j)} \left\| v_j^+ \mathrm{e}^{2\mathrm{i}k\cdot} - P \right\|_{L^2(L_j - x_{\tilde{n}_j}, L_j)}$$
$$= \frac{1}{\sqrt{k}} \inf_{P \in \mathcal{P}_{p_j,n}(0,kL_j)} \left\| v_j^+(\cdot/k)\mathrm{e}^{2\mathrm{i}\cdot} - P \right\|_{L^2(k(L_j - x_{\tilde{n}_j}), kL_j)}$$
$$\leq \sqrt{x_{\tilde{n}_j}} \inf_{P \in \mathcal{P}_{p_j,n}(0,kL_j)} \left\| v_j^+(\cdot/k)\mathrm{e}^{2\mathrm{i}\cdot} - P \right\|_{L^\infty(k(L_j - x_{\tilde{n}_j}), kL_j)}.$$

To bound this term we define the open ellipse by $E := \{w \in C : |w - k(L_j - x_n e_j)| + |^w - kL_j| < {}^R\}$ with $R := (1 + \varrho_j)kx_n e_j$ and appeal to [35, Lemma A.2] to obtain

$$\inf_{w_N \in \overline{V}_j^-} \left\| v_j^+ \mathrm{e}^{\mathrm{i}k\cdot} - w_N \right\|_{L^2(L_j - x_{\tilde{n}_j}, L_j)} \leq \sqrt{x_{\tilde{n}_j}} \frac{2}{\rho - 1} \rho^{-p_j} \left\| v_j^+(\cdot/k)\mathrm{e}^{2\mathrm{i}\cdot} \right\|_{L^\infty(\mathcal{E})}$$
$$\leq \sqrt{x_{\tilde{n}_j}} \frac{2}{\rho - 1} \rho^{-p_j} \left\| \mathrm{e}^{2\mathrm{i}\cdot} \right\|_{L^\infty(\mathcal{E})} \left\| v_j^+(\cdot/k) \right\|_{L^\infty(\mathcal{E})},$$

where $\rho := (R + \sqrt{R^2 - (kx_{\tilde{n}_j})^2})/kx_{\tilde{n}_j} = 1 + \epsilon_j + \epsilon_j^{1/2}(2 + \epsilon_j)^{1/2}$. Noting that

$$\inf_{w_N \in V_j^-} \left\| v_j^+ \mathrm{e}^{\mathrm{i}k\cdot} - w_N \right\|_{L^2(L_j - x_{\tilde{n}_j}, L_j)}$$
$$\leq \sqrt{x_{\tilde{n}_j}} \frac{2}{\rho - 1} \rho^{-p_j} \mathrm{e}^{2\alpha_j \pi \sqrt{\epsilon_j(\epsilon_j+2)}} C_j$$
$$\sup\{|\operatorname{Im}(w)| : w \in \mathcal{E}\} = \tfrac{1}{2}\sqrt{R^2 - x_{\tilde{n}_j}^2} = \tfrac{1}{2} k x_{\tilde{n}_j} \sqrt{\epsilon_j(\epsilon_j + 2)} \leq \alpha_j \pi \sqrt{\epsilon_j(\epsilon_j} \quad + \quad 2)$$

and

$\inf\{\mathrm{Re}(w) : w \in E\} = kL_j - kx_n e_j(1 + \varrho_j/2)$, it follows from Assumption 2.4 that

$$^+ M(u)k \left| k\left(L_j - x_{\tilde{n}_j}(1 + \epsilon_j/2)\right) \right|^{-\delta_j^+}$$

from which the result follows, recalling that $x_{\tilde{n}} \leq \alpha_j 2\pi/k$. □

In this proof we have approximated $v_j^+(\cdot/k)e^{2i\cdot}$ over n_j small elements with a single polynomial of degree p_j; sharper estimates may be derived along the lines of the proofe of [35, Theorem 5.2], which admits different polynomials in each element. A slightly sharper bound may be achievable by choosing non-oscillating functions √_____

on the elements close to the corners, yielding the constant $e^{\alpha j \pi \varrho j(\varrho j+2)}$ in place of √_____

$e^{2\alpha j \pi \varrho j(\varrho j+2)}$. However this would also require separate bounds close to the singularity of $v_j^{\pm}$, we would be unable to use the bounds of [35, Theorem 5.2], making the proof more complex. If α_j is chosen independently of k and sufficiently small, then the first and the last n_j elements of $Mn(0,L_j)$ are smaller than a given fraction of the wavelength of the problem.e

Construction of the stiffness matrix will have approximately similar CPU time for the single- and overlapping-mesh approach (assuming that the same quadrature routine is used in each method) for similar degrees of freedom, because the inner products are very similar. Provided α_j is chosen correctly, we expect the implementation of the single-mesh to be advantageous because implementation is easier, as there are fewer types of inner products that need computing for the Galerkin method and it is easily adapted from a standard hp solver, which would not generally be defined over an overlapping mesh. Secondly, controlling the number of degrees of freedom of the discrete space enables us to control the conditioning of the discrete system. Figure 2.4 suggests that the conditioning remains stable as the number of degrees of freedom increases, for $\alpha_j = \min\{(\mathbf{p})_i/2,2\}$ on the ith mesh element, for j = 1,2,3, whilst the conditioning for the overlapping-mesh space appears to grow exponentially for a similar number of degrees of freedom. It should be noted that the value of α_j is larger here than that which is used to produce the numerical results in §3.2.3 and §4.5, and we would expect a larger value of α_j to produce less accurate results. Hence an advantage of the single-mesh approach is that the user has control over the conditioning of the discrete system (by tweaking the parameter

α_j), but this may come at a cost (a loss of accuracy). Indeed, Figure 3.11 suggests that the condition number for $\alpha_j = \min\{(\mathbf{p})_i/8,2\}$ grows exponentially, whilst Figure 4.4 suggests that the larger choice $\alpha_j = \min\{(\mathbf{p})_i/4,2\}$ can lead to inaccurate results.

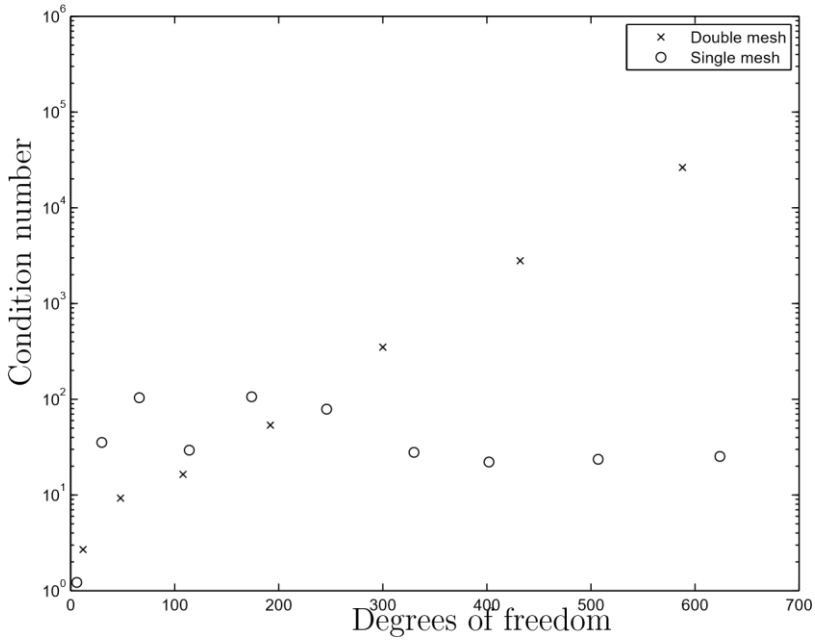


Figure 1.4: The condition number of the Galerkin stiffness matrix corresponding to scattering by a single regular triangle, with each side length 2π and wavenumber $k = 5$, for a single- and double-mesh discretisation as described in Remark 2.8. The maximal polynomial degrees used are $p = 0,...,9$ for the single-mesh space and $p = 1,...,7$ for the overlapping-mesh space; the polynomial degrees in the small elements are decreased as in Remark 2.10.

REMARK 1.10

It is shown in [35, Theorem A.3] for $V_N(\Gamma)$ that it is possible to reduce the number of degrees of freedom on Γ, whilst maintaining exponential convergence, by reducing the polynomial degree in the smaller mesh elements. For example, given $p > 1$, suppose that we define for each side Γ_j, $j = 1,...,n_\Gamma$, a degree vector $\mathbf{p}$ by

$$(\mathbf{p})_i := \begin{cases} \left\lceil \frac{i-1}{n_*} p \right\rceil, & 1 \le i \le n_*, \\ p, & n_* + 1 \le i \le n, \end{cases}$$

where n_* is the largest $i \in \{1,...,n-1\}$ such that $x_{i-1}/2 < 1$. Numerical experiments in §3.2.3 and §4.5 suggest that a similar result holds for $\overline{V}_N^{\mathrm{HNA}}(\Gamma)$, although we do not prove this here. To this purpose, one has to use the technique of [35, Theorem A.3], relying on different approximation bounds on each mesh element, while the technique used in the proof of Theorem 2.9 requires the presence of high-order polynomials on each element.

Now we present a result which compares the best approximation of the single-mesh and overlapping-mesh spaces, over the whole boundary Γ.

COROLLARY 1.11

If Assumption 2.4 and the conditions for Theorem 2.9 hold for a convex polygon Γ, then we have the following best approximation bound for the diffracted wave v_Γ (see (2.6)):

$$\inf_{w_N \in V_N^{\mathrm{HNA}}(\Gamma)} \|v_\Gamma - w_N\|_{L^2(\Gamma)} \le C_\Gamma M(u) k^{-1/2} J(k) \mathrm{e}^{-p\tau_\Gamma}, \qquad (1.10)$$

where C_Γ is a constant independent of k and

J(k) :=

(1 + kL)$^{1/2-\delta_*}$ + log$^{1/2}$(2 + kL), ((1+ kL_*)$^{1/2-\delta_*}$ + log$^{1/2}$(2+ kL_*)+ $\sqrt{k}(kI_*)^{-\delta_*}$, $V_N^{\mathrm{HNA}}(\Gamma) = \overline{V}_N^{\mathrm{HNA}}(\Gamma)$

$V^{\mathrm{HNA}}(\Gamma) = V(\Gamma)$

with $I_* := \min_j\{I_j\}$, $p := \min_j\{p_j\}$, $\tau_\Gamma := \min_{j,\pm}\{\tau_j^{\pm} p_j\}/p$, $\delta_* := \min_{j,\pm}\{\delta_j^{\pm}\}$. For HNA the case $\overline{V}_N^{\mathrm{HNA}}(\Gamma)$, it follows that $C_\Gamma = \max_j\{C_j\}$. For the case $V_N(\Gamma)$, C_Γ is equal to the constant C_4 of [35, Theorem 5.5].

Proof. For $\overline{V}_N^{\mathrm{HNA}}(\Gamma)$, the result follows by extending Theorem 2.9 to all sides, noting HNA the scaling of (2.6). The case $V_N(\Gamma)$ is proved in [35, Theorem 5.5]. □

In the above Corollary 1.11, we note that there is an additional term in the best approximation error of $\overline{V}_N^{\mathrm{HNA}}(\Gamma)$, namely $\sqrt{k}(kI_*)^{-\delta_*}$. By definition, for convex polygons we have that $\delta_* > 1/2$, hence it follows that $J(k) \sim \log^{1/2} k$ as $k \to \infty$.

REMARK 1.12

(Collocation vs Galerkin HNA BEM)**.** The collocation (see (1.19)) HNA method for the problem of the two-dimensional screen was investigated in [47]. A key component of the investigation was a comparison between single- and overlappingmesh collocation, although no conclusions were drawn regarding how the collocation points should be chosen when an overlapping mesh is used. In certain cases, choosing Chebyshev nodes resulted in a non-convergent numerical method. Given this result, HNA the single-mesh space $V_N(\Gamma)$ would be the recommended choice when coupled with a collocation HNA method. The key advantage of collocation is that the integrals in the discrete system are of one dimension less, making for easier implementation, and faster run time. The disadvantage of collocation is that there is (currently) no applicable theory on stability constants for collocation HNA BEM, however there are many problems (for example, looking ahead to multiple obstacle HNA BEM (4.22)) for which there are also no stability bounds available for the Galerkin method (see (1.18)) either, hence case there is no theoretical advantage to implementing the more complex Galerkin method. Stability analysis of collocation HNA BEM is a possible area for future work, as it has been proved (see for example [2]) for certain problems (not including HNA BEM), that collocation and Galerkin methods produce the same result, in which case the same error analysis can be applied. In summary, there are advantages and disadvantages to Collocation and Galerkin methods, depending on implementation time available and desirability of a priori error bounds.

1.3 HNA Galerkin method for a single convex polygon

Here we summarise the HNA Galerkin method for a single convex polygon Γ. We write $V_N^{\mathrm{HNA}}(\Gamma)$ to denote either of the two approximation spaces of §2.2. The continuous BIE to solve may be written by combining (1.2), (1.6) and (1.14) with either choice of A,

$$\mathcal{A} v_\Gamma = \frac{1}{k}(f - \mathcal{A}\Psi), \quad \text{on } \Gamma,$$

and the Galerkin problem (see (1.18)) to solve is: find $v_N \in V_N^{\mathrm{HNA}}(\Gamma)$ such that

$$(\mathcal{A} v_N, \varphi)_{L^2(\Gamma)} = \frac{1}{k}(f - \mathcal{A}\Psi, \varphi)_{L^2(\Gamma)}, \quad \text{for all } \phi \in V_N^{\mathrm{HNA}}(\Gamma), \tag{1.11}$$

where $(\cdot,\cdot)_{L2(\Gamma)}$ denotes the L^2 inner product on Γ (see §A.2). For A = Ak,η it follows by [16, Theorem 5.2] that there exists an $N_0 > 0$ such that (2.11) has a unique solution for $N \geq N_0$. For A = Ak we have $N_0 = 1$ (see Remark 2.13 for details). Provided the conditions of Assumption 2.4 and Corollary 2.11 are satisfied, we may bound the error in our approximation on a single convex scatterer, $\|v_\Gamma - v_N\|_{L2(\Gamma)} \leq Ck^{-1/2} C_q(k) M(u) J(k) e^{-p\tau\Gamma}$, for $N \geq N_0$, (1.12)

where C, p, J and τ_Γ are as in Corollary 2.11. The bound (1.12) follows by combining the best approximation error with [16, Theorem 5.2], and is a natural generalisation of [35, (5.11)] to multiple approximation spaces (recalling that J depends on the choice of approximation space). The constant $C_q(k) \geq 1$ denotes the stability or quasi-optimality constant, which determines the relationship between the best approximation available in the space $V_N^{HNA}(\Gamma)$ and the solution to the Galerkin equations (1.11), as these need not be the same thing. The stability constant depends on the formulation used, and has been the subject of much investigation, which we will summarise now:

REMARK 1.13

(The stability constant C_q for HNA methods on convex polygons)**.** Here C_q represents the quasi-optimality constant associated to the Galerkin method. If $A = Ak$ (of Definition 1.3) then $N_0 = 1$ and $C_q \,.\, k^{1/2}$ for $k \geq k_0$, where $k_0 > 0$ is fixed and independent of k. This follows from the coercivity of Ak, by C´ea's lemma (see [52] and [35, Theorem 6.1]). Numerical experiments of [8, Table 6.1] show that Ak,k is coercive for the square and equilateral triangle, with a coercivity constant uniform in k, hence $C_q \sim \|Ak,k\|\|k\|$; given the current best available bounds on Ak,η ([20, Theorem 1.4]), we have $C_q \,.\, k^{1/4} \log k$ in these particular cases. Such a result has not yet been proved for Ak,η with general convex polygonal scatterers. Instead we may use the more general theory (see for example [3, Theorem 3.1.1]), given that Ak,η is a compact perturbation of a coercive operator [16, p620] we have existence of N_0 and C_q, although this provides no mechanism to bound either in terms of known parameters.

THEOREM 1.14.

Suppose that the stability constant $C_q(k)$ grows at most algebraically with k, that Assumption 2.4 holds, A is a compact perturbation of a coercive operator, and either approximation space is used with the following condition on the polynomial degree vector on the jth side:

$$(\mathbf{p}_j)_i = p_j \leq n_j/c_j \quad \text{for} \quad i = 1,\ldots,n_j \quad \text{with} \quad c_j \geq 1 \quad \text{for} \quad j = 1,\ldots n_\Gamma,$$

where c_j is the constant from Theorem 2.9. Then the solution of the Galerkin equations (1.11) converges exponentially to the true solution of (1.14) as $p \to \infty$ on $L^2(\Gamma)$, where $p := \min_j\{p_j\}$.

Proof. It follows by Assumption 1.4 and Corollary 1.11 that $C_q(k)M(u)J(k)$ grows only algebraically in k. Hence there exists an $N \geq N_0$, where N grows with p, such that $e^{-p\tau\Gamma}$ will dominate the algebraic terms, given the bound (2.12). □

The above theorem highlights the importance of obtaining algebraic bounds on M(u) (thus showing that Assumption 1.4 holds), which is a key component . Although much work has been done to understand the growth of the stability constant $C_q(k)$ for star-shaped Ω_- (as discussed in Remark (1.13)), This provides the first bound for multiple scattering problems. Given the solution of (1.11), we can approximate quantities of practical interest using the Definitions (1.2) and (1.6), noting that v_N is an approximation to v_Γ, to obtain

$$\partial_{\mathbf{n}}^{+} u \approx \nu_p := \Psi + k v_N, \qquad \text{on } \Gamma,$$

where p is the polynomial degree used to obtain the approximation with N degrees of freedom (there are parameters other than p, in particular c_j, which determine N). From this, an approximation to the total field u of (1.10) follows immediately, by inserting into (1.10)

$$u \approx u_N := u^{inc} - S_k v_p, \qquad \text{in } \Omega_+,$$

and we have that

$\|u - u_N\|_{L^\infty(\Omega_+)} \leq CM(u)k^{1/2} \log^{1/2}(2 + L_*k)J(k)e^{-p\tau\Gamma}$.

Likewise, we have the following approximation to the far-field coefficient of (1.16)

$$\theta \in [0,2\pi), \qquad (1.13)$$

$$u^\infty(\theta) \approx u_N^\infty(\theta) := \int_\Gamma e^{-ik[y_1 \cos\theta + y_2 \sin\theta]} v_p(\mathbf{y})\, ds(\mathbf{y}),$$

with error estimate (from [35, Theorem 6.4])

$$\|u^\infty - u_N^\infty\|_{L^\infty(0,2\pi)} \leq Ck^{3/2} M(u) \log^{1/2}(2 + kL_*) J(k) e^{-p\tau_\Gamma}. \qquad (1.14)$$

II. CONCLUSION

The investigation was a comparison between single- and overlappingmesh collocation, although no conclusions were drawn regarding how the collocation points should be chosen when an overlapping mesh is used. In certain cases, choosing Chebyshev nodes resulted in a non-convergent numerical method. Given this result, HNA the single-mesh space $V_N(\Gamma)$ would be the recommended choice when coupled with a collocation HNA method. The key advantage of collocation is that the integrals in the discrete system are of one dimension less, making for easier implementation, and faster run time. The disadvantage of collocation is that there is (currently) no applicable theory on stability constants for collocation HNA BEM, however there are many problems (for example, looking ahead to multiple obstacle HNA BEM for which there are also no stability bounds available for the Galerkin method (see (1.18)) either, hence case there is no theoretical advantage to implementing the more complex Galerkin method. Stability analysis of collocation HNA BEM is a possible area for future work, as it has been proved for certain problems (not including HNA BEM), that collocation and Galerkin methods produce the same result, in which case the same error analysis can be applied. In summary, there are advantages and disadvantages to Collocation and Galerkin methods, depending on implementation time available and desirability of a priori error bounds.

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